

Effective actions on squashed lens spaces

Marcelo De Francia*, Klaus Kirsten† and J.S.Dowker ‡

*Department of Theoretical Physics,
The University of Manchester, Manchester, England*

Abstract

As a technical exercise with possible relevance to the holographic principle and string theory, the effective actions (functional determinants) for scalars and spinors on the squashed three-sphere identified under the action of a cyclic group, \mathbb{Z}_m , are determined. Especially in the extreme oblate squashing limit, which has a thermodynamic interpretation, the high temperature behaviour is found as a function of m . Although the intermediate details for odd and even m are different, the final answers are the same. A thermodynamic interpretation for spinors is possible only for twisted periodicity conditions and m even.

*dfrancia@a13.ph.man.ac.uk

†klaus@a13.ph.man.ac.uk

‡dowker@a13.ph.man.ac.uk

1. Introduction

Quantum field theory on the squashed three-sphere is of current interest in connection with the holographic principle but has an intrinsic technical value and this is the aspect that we mainly wish to promote here. The present work amends and extends earlier work of the last named author. In particular, attention is drawn to some errors of calculation in [1] which seriously affect the conclusions there. Especially, the overall sign of the spinor free energy is reversed, making a high temperature thermodynamic interpretation impossible. This could have been anticipated on the grounds that on the simply-connected squashed three-sphere, \tilde{S}^3 , the spinor field will be single-valued, and hence periodic around the fiber ψ -circle while a thermal interpretation would require anti-periodicity. This ties in with the topological fact that the bulk space-time is not spin. In order to define a conventional spin structure it is sufficient to identify the bulk spacetime for example by quotienting the ψ -circle by the cyclic group \mathbb{Z}_m with m even. We write this as \tilde{S}^3/\mathbb{Z}_m and refer to it as a squashed lens space. Once the space is multiply connected there is the allowed possibility of twisted, and, in particular, of anti-periodic fields so that a thermal interpretation for spinors becomes possible. On the squashed lens space this can occur only for even m , which again fits in nicely with the absence of spinor structures in the bulk for odd m . In fact we find that odd m is technically more awkward than even m , although the final answers are identical.

In this paper we shall be particularly concerned to exhibit the explicit high-temperature behaviours of the free energies in the thermodynamic interpretation and their dependence on the identification parameter, m .

2. Fields on the squashed lens space

In this section we concentrate on the mode description, and thence the ζ -functions, $\zeta(s)$, for scalar and spinor fields on squashed lens spaces but, first, in order to make this paper relatively self contained, the geometry of the squashed three-sphere is briefly outlined, see [1,2] and references there. The local geometry of \tilde{S}^3 is determined by the metric

$$ds^2 = (d\theta^2 + \sin^2 \theta d\phi^2) + l_3^2 (d\psi + \cos \theta d\phi)^2.$$

Oblate deformations correspond to $l_3 < 1$ and prolate to $l_3 > 1$. When the squashing parameter l_3 equals one, the metric is that of a round three-sphere, S^3 , of radius 2, expressed in Euler angles.

In the unidentified case, the ψ -circle has a circumference of $4\pi l_3$. For the lens space, with m identifications around this circle, the periodicity is $4\pi l_3/m$ and, in the thermodynamical interpretation, where ψ plays the role of a Euclidean time (roughly speaking), this periodicity translates to an inverse temperature $\beta = 4\pi l_3/m$. We therefore *define* a free energy $\Phi(\beta)$ to equal $\mp \zeta'(0)/(2\beta)$, $-$ for scalars and $+$ for spinors, and shall seek to compare this, roughly, with a free energy on a 2-sphere to which the metric tends as β becomes small.

Scalar field

We recall that, up to normalisation, the scalar modes on \tilde{S}^3 are the $SU(2)$ representation matrices

$$\begin{aligned} \mathcal{D}_{NM}^{(L)}(\phi, \theta, \psi) &= e^{iN\phi} P_{NM}^{(L)}(\theta) e^{iM\psi}, \\ (L = 0, 1/2, 1, \dots, \quad -L \leq N, M \leq L) \end{aligned} \quad (2.1)$$

with eigenvalues

$$\lambda_{LM} = \frac{(2L+1)^2}{4} - l_3^{-2} \delta^2 M^2 \quad (2.2)$$

for the operator $-\Delta + 1/4$, where $\delta^2 = l_3^2 - 1$ has been introduced. The degeneracy is $n = 2L + 1$.

Working with a second order differential propagation operator for the scalar field means that there is considerable latitude in its choice, in the sense that various covariant quantities can be added more or less naturally. The most usual addition is a term proportional to the scalar curvature, R . The choice made here is the one used in [1], the sole reason being the relative simplicity of the eigenvalues. Other selections were briefly considered in [1]. If one were seriously concerned about conformal invariance in three dimensions then the relevant operator would be $-\Delta + R/8$ for which the eigenvalues are

$$\lambda_{LM}^C = \lambda_{LM} - \frac{l_3^2}{16}. \quad (2.3)$$

Results will be presented for both sets of eigenvalues.

The lens space factoring, \tilde{S}^3/\mathbb{Z}_m , is implemented by the identification of ψ and $\psi + 4\pi h/m$, $h = 1, 2, \dots, m-1$, which entails, via periodicity, the restriction

$$\frac{2M}{m} = k, \quad k \in \mathbb{Z}. \quad (2.4)$$

A generalisation is afforded by the field pseudo-periodicity

$$\psi(q\gamma) = a(\gamma)\psi(q), \quad q = (\phi, \theta, \psi), \quad \gamma \in \mathbb{Z}_m \quad (2.5)$$

where $a(\gamma)$ is a unitary one-dimensional representation of \mathbb{Z}_m ,

$$a(\gamma_0^h) = e^{2\pi i h r / m}, \quad h = 0, 1, \dots, m-1,$$

labelled by $r = 0, 1, \dots, m-1$. The action of the generator, γ_0 , of \mathbb{Z}_m on q is $q\gamma_0 = (\phi, \theta, \psi + 4\pi/m)$. Equation (2.5) leads to the twisting restriction

$$\frac{2M - r}{m} = k \quad (2.6)$$

instead of (2.4).

The simplest example is $m = 2$. When $r = 0$, M , and therefore L , is integral giving periodic fields while when $r = 1$, L is half-odd integral giving anti-periodic, or ‘twisted’, fields. In the present context we might call the former ‘thermal’ and the latter ‘anti-thermal’ and we shall restrict the discussion in this paper to these cases. Anti-thermal bosons mimic thermal fermions.

In the general case when m is even these two are the only possibilities for real fields and correspond to $r = 0$ and $r = m/2$. When m is odd, there are no real twisted fields and, furthermore, the corresponding bulk space-time is not spin. For this reason we separate even and odd m and begin with the even and periodic ($r = 0$) case.

The Mode Lattice

Rather than construct projections we deal with the consequences of the restriction (2.4) directly. It is convenient to represent the modes in the *unidentified* case on a square lattice and then impose any identification restrictions geometrically. The overall restriction $|2M| < 2L$ is implemented by confining the lattice to the positive quadrant, choosing the horizontal axis, x , to equal $L - M$ and the vertical one, y , to be $L + M$. Thus the *diagonal axis* is $x + y = 2L = n - 1$ and the *anti-diagonal* one is $y - x = 2M$ and has a finite range. Unidentified modes correspond to *every* integer lattice point in the positive x, y quadrant, including the edges.

The condition (2.4) is then allowed for by picking only those square lattice points that coincide with the intersections of the line $n = \text{constant}$ (for each $n = 1, 2, \dots$ in turn) with the 45° lines, $2M = km, \forall k \in \mathbb{Z}$. This is best done for a few m and the general rule inferred. In this way we come to the residue class mod m decomposition of the odd number $n = 2L + 1$,

$$n = pm + (2\nu + 1), \quad p = 0, 1, \dots, \infty, \quad \nu = 0, 1, \dots, m/2 - 1, \quad (2.7)$$

which encompasses all (positive) odd numbers.

It is then easy to see that the limits on M imply that, for a given p , $|k| \leq p$, [3]. In order to maintain contact with the development of our previous work, it is better to write this as $0 \leq u \leq 2p$, where $u = p - k$ so that the scalar ζ -function reads

$$\zeta^{(e)}(s) = m \left(\frac{2l_3}{m} \right)^{2s} \sum_{\nu=0}^{m/2-1} \sum_{p=0}^{\infty} \sum_{u=0}^{2p} \frac{p + \bar{\nu}/m}{[(p + \bar{\nu}/m)^2 + \delta^2(u + \bar{\nu}/m)(2p - u + \bar{\nu}/m)]^s} \quad (2.8)$$

where we have set $\bar{\nu} = 2\nu + 1$ for brevity.

Equivalent to these label relations is the residue class decomposition, $L - M = u(m/2) + \nu$ where u plays the role of q in our earlier computations, [1,2].

The case of *odd* m seems to present some peculiar technical difficulties. Equation (2.4) means that M , and hence L , can be integer or half odd-integer so that n can be both even and odd but with some even omissions. Again, with reference to the mode lattice, decompose $n = 2L + 1$,

$$n = pm + \bar{\nu}, \quad p = 0, 1, \dots, \infty, \quad \bar{\nu} = 2\nu + 1, \quad \nu = 0, 1, \dots, m - 1. \quad (2.9)$$

Note that n runs over all the integers *except for* the evens, $2, 4, \dots, m - 1$.

The same limits apply as before and defining $q = p/2 - M/m$ we have $0 \leq q \leq p$. Equivalently, $L - M = qm + \nu$, which, as a check, reduces to q when $m = 1$.

The ζ -function is

$$\zeta^{(o)}(s) = m \left(\frac{2l_3}{m} \right)^{2s} \sum_{\nu=0}^{m-1} \sum_{p=0}^{\infty} \sum_{q=0}^p \frac{p + \bar{\nu}/m}{[(p + \bar{\nu}/m)^2 + 4\delta^2(q + \bar{\nu}/2m)(p - q + \bar{\nu}/2m)]^s} \quad (2.10)$$

and the calculation is essentially identical, in structure at least, to the unidentified case. When $m = 1$ we just set $n = p + 1$ to regain the scalar expressions in our earlier paper.

The Dirac field

Before analysing the ζ -functions (2.8), (2.10), we turn to the spin-half field.

Apart from a possible mass term there is little one can add to the Dirac operator, in a natural way. To recall, the unidentified eigenvalues of the Weyl operator are

$$\omega_{\pm} = \frac{1}{2} \left[\frac{1}{2} l_3 \pm [(2L + 1)^2 - 4M^2 l_3^{-2} \delta^2]^{1/2} \right]$$

with $-(L \pm 1/2) \leq M \leq L \pm 1/2$ and $L \geq 0$ for ω_+ , $L \geq 1/2$ for ω_- .

Some relevant analysis in the identified case has been performed by Gibbons, Pope and Römer, [4] and Pope, [5] in connection with boundary terms in the index theorem. However in these calculations only the two ‘extreme’ modes were relevant. (See also [6].) The more recent mathematical investigations by Bär, [7,8], detail the spectrum and also that on the squashed odd sphere, S^{2n+1} , which is referred to as the Berger sphere.

On \widetilde{S}^3 the same restriction holds as in the scalar case, that is, for periodic spinors

$$2M = 0 \bmod m \quad (2.11)$$

and the calculation can proceed, more or less, as before.

There are now positive and negative modes. For the former, the orbital quantum number L satisfies $L \geq 0$ while for the latter we have $L \geq 1/2$ and the corresponding ranges for the projection number M are $|M| \leq L + 1/2$ and $|M| \leq L - 1/2$ respectively. We consider the effect of the restriction (2.11) on the two sets of modes separately and again begin with even m when we can easily see that $n = 2L + 1$ must also be even.

For the positive modes we decompose n as

$$n = pm + 2\nu, \quad p = 0, 1, \dots, \infty, \quad \nu = 0, 1, \dots, m/2 - 1. \quad (2.12)$$

Note that this would lead to the inclusion of the value $n = 0$ which is not within the mode labelling scheme. However the degeneracy factor of n allows an extension to this value. Note also that all even numbers occur.

Referring to the mode lattice, for the positive modes the diagonal label is n , which takes the value 0 at the origin. The ‘antidiagonal’ label is $2M$. It is then easy to see that (2.11) is equivalent to $|2M/m| \leq p$. For the negative modes the diagonal label starts at the origin with the value $n = 2$. In this case we use the decomposition

$$n = pm + 2\nu', \quad p = 0, 1, \dots, \infty, \quad \nu' = 1, 2, \dots, m/2, \quad (2.13)$$

and, defining $u = p - 2M/m$, the relevant summations read

$$\zeta_{\pm}^{(e)}(s) = \left(\frac{2l_3}{m}\right)^{2s} \sum_{\nu=\nu_{\pm}}^{m_{\pm}} \sum_{p=0}^{\infty} \sum_{u=0}^{2p} \frac{mp + 2\nu}{\left[\left(p + \frac{2\nu}{m}\right)^2 + \delta^2\left(u + \frac{2\nu}{m}\right)\left(2p - u + \frac{2\nu}{m}\right)\right]^{1/2} \pm \frac{l_3^2}{2m}}^{2s} \quad (2.14)$$

where $m_+ = m/2 - 1$, $m_- = m/2$, $\nu_+ = 0$ and $\nu_- = 1$, (+ referring to the positive modes and - to the negative).

After expanding in $l_3^2/(2m)$, in order to obtain some cancellation between the positive and negative modes, the differing summations over the roots of unity have to be adjusted. In particular in ζ_+ we add the term $\nu = m/2$ and subtract the term $\nu = 0$ in order to convert the summation labels into those for ζ_- .

After some algebra, combining ζ_+ and ζ_- , it is then seen, expanding in $l_3^2/(2m)$, that the calculation will devolve upon evaluating, and continuing, the expression

$$f_m^{(e)}(s) = \sum_{\nu=1}^{m/2} \sum_{p=0}^{\infty} \sum_{u=0}^{2p} \frac{mp + 2\nu}{[(p + 2\nu/m)^2 + \delta^2(u + 2\nu/m)(2p - u + 2\nu/m)]^s} \quad (2.15)$$

to be compared with (2.8). The precise relation for $\zeta^{(e)}(s) = \zeta_+^{(e)}(s) + \zeta_-^{(e)}(s)$ is

$$\begin{aligned} \zeta^{(e)}(s) = 2 \left(\frac{2l_3}{m} \right)^{2s} & \left(m\zeta_H(2s-1, l_3^2/2m) - \frac{l_3^2}{2} \zeta_H(2s, l_3^2/2m) + f_m^{(e)}(s) \right. \\ & \left. + \frac{l_3^4}{4m^2} s(2s+1) f_m^{(e)}(s+1) + \dots \right), \end{aligned} \quad (2.16)$$

the first two terms coming from the aforementioned adjustment.

For odd m we again consider the positive and negative modes separately. For the former, reference to the lattice shows that with $n = mp + 2\nu$, $\nu = 0, 1, \dots, m-1$ we have $|M/m| \leq p/2$, while for the latter $n = mp + 2\nu$ with $\nu = 1, 2, \dots, m$ and the same condition on M . Hence, defining $q = p/2 - M/m$ we have $0 \leq q \leq p$ (as in the scalar case) and the ζ_- functions are

$$\zeta_{\pm}^{(o)}(s) = \left(\frac{2l_3}{m} \right)^{2s} \sum_{\nu=\nu_{\pm}}^{m_{\pm}} \sum_{p=0}^{\infty} \sum_{q=0}^p \frac{mp + 2\nu}{[(p + \frac{2\nu}{m})^2 + 4\delta^2(q + \frac{\nu}{m})(p - q + \frac{\nu}{m})]^{1/2} \pm \frac{l_3^2}{2m}}^s \quad (2.17)$$

where now $m_+ = m - 1$ and $m_- = m$.

Similar manipulations as before produce the same expansion,

$$\begin{aligned} \zeta^{(o)}(s) = 2 \left(\frac{2l_3}{m} \right)^{2s} & \left(m\zeta_H(2s-1, l_3^2/2m) - \frac{l_3^2}{2} \zeta_H(2s, l_3^2/2m) + f_m^{(o)}(s) \right. \\ & \left. + \frac{l_3^4}{4m^2} s(2s+1) f_m^{(o)}(s+1) + \dots \right), \end{aligned} \quad (2.18)$$

where $f_m^{(o)}(s)$ is

$$f_m^{(o)}(s) = \sum_{\nu=1}^m \sum_{p=0}^{\infty} \sum_{q=0}^p \frac{mp + 2\nu}{[(p + 2\nu/m)^2 + 4\delta^2(q + \nu/m)(p - q + \nu/m)]^s}. \quad (2.19)$$

Conformal scalars

Conformal scalars are similar to spinors in that an expansion in l_3^2 is required. The form of the eigenvalues (2.3) shows that the only difference from the small l_3 series of the effective action will be in the l_3^2 term. Denoting by $\zeta^C(s)$ the ζ -function associated with the spectrum (2.3), the relation between the ζ -functions is

$$\zeta^C(s) = \zeta(s) + \frac{1}{16}sl_3^2\zeta(s+1) + O(l_3^4)$$

so that the effective actions are connected by

$$\zeta^{C'}(0) = \zeta'(0) + \frac{1}{16}l_3^2\zeta(1) + O(l_3^4).$$

Note that $\zeta(1)$ is finite because we are dealing with a manifold without boundary so that $\zeta^{C'}(0)$ is well defined.

Twisted spinors

So far, for both scalars and spinors, only periodic fields have been analysed. As discussed in the introduction, we are interested in the case of anti-periodic spin-half fields. That is, those fields that pick up a factor of -1 on going once round the ψ -circle on \tilde{S}^3/\mathbb{Z}_m . As outlined in section 2 this corresponds to the choice of $r = m/2$, with m even, for the integer, r , that determines the twisting.

The general condition on the modes leads to the restriction $2M = km + r$ or, in the anti-periodic case

$$2M = (2k+1)\frac{m}{2}, \quad k \in \mathbb{Z}$$

a simple consequence of which is that M can never equal zero.

A further brief subdivision of the order m is required. Even m 's are either such that m is divisible by 4 (*i.e.* $m/2$ is even) or such that $m-2$ is divisible by 4 (*i.e.* $m/2$ is odd). For the first case $2M$ is odd while for the latter $2M$ is even. The two cases are initially treated separately but reference to the mode lattice soon produces the same mode counting. This shows that for the positive modes

$$n = (p + 1/2)m + 2\nu, \quad p = 0, 1, \dots, \infty, \quad \nu = 0, 1, \dots, m/2 - 1.$$

The negative modes have an n shifted by 2,

$$n = (p + 1/2)m + 2\nu, \quad p = 0, 1, \dots, \infty, \quad \nu = 1, 2, \dots, m/2.$$

We note that when $m/2$ is odd (even), n is odd (even).

The restriction on M is $|2M/m| \leq p + 1/2$ in both cases so that, defining $u = p + 1/2 - 2M/m$, one has $0 \leq u \leq 2p + 1$ and the ζ -functions are

$$\zeta_{\pm}^{(ap)}(s) = \left(\frac{2l_3}{m}\right)^{2s} \sum_{\nu=\nu_{\pm}}^{m_{\pm}} \sum_{p=0}^{\infty} \sum_{u=0}^{\bar{p}} \frac{m\bar{p}/2 + 2\nu}{\left[\left(\frac{\bar{p}}{2} + \frac{2\nu}{m}\right)^2 + \delta^2\left(u + \frac{2\nu}{m}\right)(\bar{p} - u + \frac{2\nu}{m})\right]^{1/2} \pm \frac{l_3^2}{2m}}^{2s} \quad (2.20)$$

where attention has been transferred to the odd integer $\bar{p} = 2p + 1$, and m_{\pm} are as in (2.14).

The expansion function is

$$f^{(ap)}(s, m) = \sum_{\nu=1}^{m/2} \sum_{\bar{p}=1,3,\dots}^{\infty} \sum_{u=0}^{\bar{p}} \frac{m\bar{p}/2 + 2\nu}{\left[(\bar{p}/2 + 2\nu/m)^2 + \delta^2(u + 2\nu/m)(\bar{p} - u + 2\nu/m)\right]^s}. \quad (2.21)$$

Looking at the \bar{p} -sum, we can use the useful identity that ‘sum over odds = sum over all – sum over evens’. Firstly, thinking of \bar{p} as ‘all’, we see that (2.20) is identical to the *odd* m expression (2.17) with, *formally*, $m \rightarrow m/2$. Secondly, taking p as ‘evens’, (2.20) is identical to the *even* m expression (2.14) with $m \rightarrow m$ and so there is no need to do any real further algebra when $m/2$ is odd, the relation being,

$$\zeta^{(ap)}(s, m) = \zeta^{(o)}(s, m/2) - \zeta^{(e)}(s, m), \quad (m/2 \text{ odd}), \quad (2.22)$$

or, in terms of the f function,

$$f^{(ap)}(s, m) = 2^{2s} f^{(o)}(s, m/2) - f^{(e)}(s, m), \quad (m/2 \text{ odd}), \quad (2.23)$$

with obvious notation. A similar conclusion holds for even $m/2$, after making some formal identifications. As we will see later, twisted spinors allow for a thermal interpretation.

3. Continuation of the ζ -functions.

We have so far mainly concentrated on setting up the mode properties in the identified case. The previous papers, [1,2], supply most of the techniques needed for the continuation of the resulting ζ -functions. The method for this, which is not unique, involves a Plana summation applied to the inner summation followed, if necessary, by a Watson-Sommerfeld procedure applied to the p -sum. A particular

arrangements of cuts in the introduced complex planes, resulting from the vanishing of denominators, allows the extraction of factors of $\sin\pi s$ and hence a practical evaluation of $\zeta'(0)$. Here we exhibit some details just in the case of the scalar field for m even. The spinor case presents only a little extra algebra and, as already mentioned, while odd m is rather more awkward in the intermediate calculation it finally yields the same m dependence.

As in Ref. [1], we seek to perform the u sum using the Plana method. Denote, therefore, the last sum in (2.8) by I and, this time, choose the relevant strip of the u -plane to be $-\frac{\bar{\nu}}{m} \leq \text{Re } u \leq 2p + \frac{\bar{\nu}}{m}$. In the oblate case, $l_3 < 1$, there are three parts to the summation formula, $I = I_1 + I_2 + I_3$. They read explicitly

$$I_1 = \int_{-\bar{\nu}/m}^{2p+\bar{\nu}/m} \frac{(p + \bar{\nu}/m)du}{[(p + \bar{\nu}/m)^2 + \delta^2(u + \bar{\nu}/m)(2p - u + \bar{\nu}/m)]^s}, \quad (3.1)$$

$$I_2 = -4 \text{Im} \int_0^\infty \frac{p + \bar{\nu}/m}{[(p + iB')^2 - C^2]^s} \frac{dx}{1 - e^{2\pi x - 2i\pi\bar{\nu}/m}}, \quad (3.2)$$

$$I_3 = \int_{C_U} \frac{du}{1 - e^{-2\pi iu}} \frac{p + \bar{\nu}/m}{[(p + \bar{\nu}/m)^2 + \delta^2(u + \bar{\nu}/m)(2p - u + \bar{\nu}/m)]^s} \\ - \int_{C_L} \frac{du}{1 - e^{2\pi iu}} \frac{p + \bar{\nu}/m}{[(p + \bar{\nu}/m)^2 + \delta^2(u + \bar{\nu}/m)(2p - u + \bar{\nu}/m)]^s}. \quad (3.3)$$

The first is the integral approximation. In the second we use

$$B' = -x\delta^2 - i\frac{\bar{\nu}}{m} = -x(l_3^2 - 1) - i\frac{\bar{\nu}}{m} = B - i\frac{\bar{\nu}}{m}, \quad C^2 = x^2(1 - l_3^2)l_3^2,$$

and the third term comes from a clockwise contour surrounding the extra cuts that appear in the u -complex plane when $l_3 < 1$. The three contributions are dealt with one by one.

Changing the integration variable in (3.1) we immediately obtain the first contribution to the ζ -function,

$$\zeta_{e,1}(s) = 2m \left(\frac{2l_3}{m}\right)^{2s} \left[\sum_{\nu=0}^{m/2-1} \sum_{p=0}^{\infty} \left(p + \frac{\bar{\nu}}{m}\right)^{-2s+2} \right] \int_0^1 \frac{dx}{(1 + 4\delta^2 x(1-x))^s}. \quad (3.4)$$

The sum in square brackets can be converted to the original n -mode sum (see eqn. (2.7)), running, in this case, over odd n -values, to give

$$\zeta_{e,1}(s) = \frac{8}{m} l_3^{2s} \zeta_H(2s - 2, 1/2) \int_0^1 \frac{dx}{(1 + 4\delta^2 x(1-x))^s}. \quad (3.5)$$

Three conclusions may be derived straightforwardly from expression (3.5). In the first place, its contribution to the conformal anomaly, $\zeta_{e,1}(0)$, vanishes. Secondly, its value at $s = 1$, needed in the evaluation of the conformal field effective action, is $O(l_3^2)$ and is negligible in our present calculation. Thirdly, the Weyl pole at $s = 3/2$ correctly has a volume factor of $1/m$.

Taking the derivative explicitly, we obtain the first contribution to the effective action,

$$\zeta'_{e,1}(0) = \frac{3}{m\pi^2} \zeta_R(3). \quad (3.6)$$

The second term in the effective action comes from (3.2). We proceed to its calculation again using the Plana technique, in this case to effect the p -sum.

Using the integral representation and changing variables

$$\sum_{p=0}^{\infty} \frac{p + \bar{\nu}/m}{[(p + iB')^2 - C^2]^s} = \frac{1}{2i} \int_{L_D} d\zeta \frac{\zeta - iB}{(\zeta^2 - C^2)^s} \cot(\pi(\zeta - iB')) \quad (3.7)$$

where L_D runs anticlockwise around the poles at $\zeta = p + iB'$, $p \in \mathbb{N}_0$, of the last factor.

The integration path can be deformed towards the imaginary axis and two contributions to $\zeta_e(s)$ appear *viz.* the integral over the vertical axis,

$$\begin{aligned} \zeta_{e,(2,1)}(s) = 2m \left(\frac{2l_3}{m} \right)^{2s} \text{Im} \sum_{\nu=0}^{m/2-1} \int_0^{\infty} \frac{dx}{1 - e^{2\pi x - 2i\pi\bar{\nu}/m}} \times \\ \left\{ e^{-i\pi s} \int_0^{\infty} dy \frac{y - B}{(y^2 + C^2)^s} \coth(\pi(y - B')) + \right. \\ \left. e^{i\pi s} \int_0^{\infty} dy \frac{y + B}{(y^2 + C^2)^s} \coth(\pi(y + B')) \right\} \end{aligned} \quad (3.8)$$

and the integration around the cut on the real axis between $-C$ and C ,

$$\begin{aligned} \zeta_{e,(2,2)}(s) = 4m \sin(\pi s) \left(\frac{2l_3}{m} \right)^{2s} \text{Im} \sum_{\nu=0}^{m/2-1} \int_0^{\infty} \frac{dx}{1 - e^{2\pi x - 2i\pi\bar{\nu}/m}} \times \\ \int_0^C dy \frac{(y - iB)}{(C^2 - y^2)^s} \cot(\pi(y - iB')). \end{aligned} \quad (3.9)$$

We begin by studying $\zeta_{e,(2,1)}(s)$. First we make the split, $\coth(\pi x) = 1 + 2/[\exp(2\pi x) - 1]$. Due to the ν -summation several contributions disappear leading to the expression

$$\zeta_{e,(2,1)}(s) = 2m \left(\frac{2l_3}{m} \right)^{2s} \sin(\pi s) \text{Re} \sum_{\nu=0}^{m/2-1} \int_0^{\infty} \frac{dx}{1 - e^{2\pi x - 2i\pi\bar{\nu}/m}} \left[\int_0^{\infty} \frac{2Bdy}{(y^2 + C^2)^s} \right]$$

$$- 2 \int_0^\infty \frac{dy}{(y^2 + C^2)^s} \left(\frac{y - B}{e^{2\pi(y-B)} e^{2\pi i \bar{\nu}/m} - 1} - \frac{y + B}{e^{2\pi(y+B)} e^{-2\pi i \bar{\nu}/m} - 1} \right) \quad (3.10)$$

As for $\zeta_{e,1}$, the ‘anomalous’ contribution of this term vanishes. Moreover, $\zeta_{e,(2,1)}(1) = O(l_3^2)$, again giving no contribution to the effective action of the conformal field to the order considered here.

Concerning the evaluation of the derivative at $s = 0$, it is not difficult to see that the first term in (3.10) does not contribute. So, we get

$$\zeta'_{e,(2,1)}(0) = -4\pi m \operatorname{Re} \sum_{\nu=0}^{m/2-1} \int_0^\infty \frac{dx}{1 - e^{2\pi x - 2\pi i \bar{\nu}/m}} \int_0^\infty dy \times \left(\frac{y - B}{e^{2\pi(y-B)} e^{2\pi i \bar{\nu}/m} - 1} - \frac{y + B}{e^{2\pi(y+B)} e^{-2\pi i \bar{\nu}/m} - 1} \right). \quad (3.11)$$

The y -integral leads to the polylogarithm function $\operatorname{Li}_s(x) = \sum_{n=1}^\infty x^n/n^s$,

$$\zeta'_{e,(2,1)}(0) = -\frac{m}{\pi} \operatorname{Re} \sum_{\nu=0}^{m/2-1} \int_0^\infty \frac{dx}{1 - e^{2\pi x - 2\pi i \bar{\nu}/m}} \left\{ \operatorname{Li}_2(e^{-2\pi i \bar{\nu}/m}) + \operatorname{Li}_2(e^{2\pi i \bar{\nu}/m}) - 2\operatorname{Li}_2(e^{2\pi i \bar{\nu}/m - 2\pi B}) - 4\pi B \operatorname{Li}_1(e^{2\pi i \bar{\nu}/m - 2\pi B}) + \pi^2 B^2 \right\}. \quad (3.12)$$

The x -integral can be performed by expanding the denominator in exponentials and afterwards the ν -summations can be done with the help of

$$\sum_{\nu=0}^{m/2-1} e^{2\pi i \bar{\nu} n/m} = \frac{m}{2} (-1)^k \delta_{n, km/2}, \quad k \in \mathbb{Z}.$$

After lengthy calculations one arrives at

$$\begin{aligned} \zeta'_{e,(2,1)}(0) = & \left(-\frac{3(\bar{B}^2 + 4)}{8m\pi^2} + \frac{m^2}{4\pi^2} \right) \zeta_R(3) + \frac{1}{6} \gamma \\ & - \frac{2}{\pi^2} \sum_{t=1}^\infty \frac{(-1)^t}{t^2} \psi(mt/2) + \frac{m^2}{\pi^2} \sum_{t=1}^\infty (-1)^t \sum_{l=1}^{mt/2-1} \frac{\bar{B}^2}{(tm - 2l)(tm + (\bar{B} - 2)l)^2}, \end{aligned} \quad (3.13)$$

where

$$\bar{B} = 2 - \frac{m^2}{8\pi^2} \beta^2$$

has been introduced. Proceeding similarly with $\zeta'_{e,(2,2)}(0)$ we arrive at

$$\begin{aligned} \zeta'_{e,(2,2)}(0) = & \frac{3\bar{C}^2}{8m\pi^2} \zeta_R(3) \\ & - \frac{m^2}{\pi^2} \sum_{n=1}^\infty (-1)^n \bar{C}^{2n} \sum_{t=1}^\infty (-1)^t \sum_{l=1}^{tm/2-1} l^{2n-2} \frac{2\bar{B}l - (tm - 2l)(2n - 1)}{((\bar{B} - 2)l + mt)^{2n+2}} \end{aligned} \quad (3.14)$$

with

$$\bar{C}^2 = \left(4 - \frac{m^2}{4\pi^2}\beta^2\right) \frac{m^2}{16\pi^2}\beta^2.$$

Finally, expansion of expressions (3.13) and (3.14) gives the high temperature expansion of both pieces of the second part to the effective action,

$$\zeta'_{e,(2,1)}(0) = \left(-\frac{3}{m\pi^2} + \frac{m^2}{4\pi^2}\right) \zeta_R(3) + \beta^2 \left(\frac{m^2}{48\pi^2} - \frac{3m}{16\pi^4}\zeta_R(3)\right) + O(\beta^4) \quad (3.15)$$

and

$$\zeta'_{e,(2,2)}(0) = \beta^2 \left(-\frac{m^2}{192\pi^2} + \frac{3m}{16\pi^4}\zeta_R(3)\right) + O(\beta^4). \quad (3.16)$$

To complete our calculation, the oblate-only contribution, (3.3), to the effective action must be evaluated. It is straightforward to see that a change of variables gives

$$\zeta_{e,3}(s) = \frac{2^{2s+1}}{m} i \tan \theta \sum_{\nu=0}^{m/2-1} \sum_{p=0}^{\infty} n^{2-2s} \int_C \frac{1}{1 - e^{2\pi\zeta \tan \theta n/m}} \frac{d\zeta}{(1 - \zeta^2)^s} \quad (3.17)$$

where the label n , see (2.7), is reintroduced, and where C is a clockwise contour enclosing the cut at $\xi \in [1, \infty)$ in the complex ξ -plane.

Again $\zeta_{e,3}(0) = 0$, but now, the conformally required term $\zeta_{e,3}(1)$ is relevant to the evaluated order β^2 . We will return to its calculation after the analysis of the derivative.

Writing (3.17) as a real integral and introducing

$$\beta' = \frac{4\pi \tan \theta}{m} = \beta \left(1 - \frac{m^2}{16\pi^2}\beta^2\right)^{-1/2} \quad (3.18)$$

we get,

$$\zeta'_{e,3}(0) = -\beta' \sum_{\nu=0}^{m/2-1} \sum_{p=0}^{\infty} n^2 \int_1^{\infty} \frac{d\eta}{1 - e^{\eta\beta'n/2}}. \quad (3.19)$$

The denominator can be given as a geometric series and considering the integral representation for the exponential,

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Gamma(s) ds, \quad \text{Re } c > 0, \quad (3.20)$$

we immediately obtain the expression

$$\zeta'_{e,3}(0) = 4 \sum_{\text{poles}} \text{Res} \left[(\beta')^{-s} \zeta_H(s-1, 1/2) \Gamma(s) \zeta_R(s+1) \right] \quad (3.21)$$

where terms of different order in β are related to successive poles in the expression in square brackets.

Then, the final result for the third contribution is

$$\begin{aligned}\zeta'_{e,3}(0) &= \frac{4}{\beta^2}\zeta_R(3) - \frac{m^2}{4\pi^2}\zeta_R(3) - \frac{1}{6}\log 2 - \frac{1}{6}\log \beta - 2\zeta'_R(-1) \\ &\quad - \frac{m^2\beta^2}{192\pi^2} + \frac{7\beta^2}{5760} + O(\beta^4).\end{aligned}\tag{3.22}$$

A similar procedure allows one to calculate the extra contribution needed for conformal fields previously mentioned,

$$\begin{aligned}\zeta_{e,3}(1) &= 2\beta' \sum_{\text{poles}} \text{Res}[(\beta')^{-s}\Gamma(s)\zeta_H(s, 1/2)\zeta_R(s)] \\ &= 2(\gamma + 2\log 2 - \log \beta) + O(\beta^2).\end{aligned}\tag{3.23}$$

Putting all the results (3.6), (3.15), (3.16), (3.22) and (3.23) together, our complete final expressions for the m -even case turn out to be,

$$\zeta'_e(0) = \frac{4}{\beta^2}\zeta_R(3) - \frac{1}{6}\log 2 - \frac{1}{6}\log \beta - 2\zeta'_R(-1) + \beta^2 \left(\frac{m^2}{96\pi^2} + \frac{7}{5760} \right) + O(\beta^4),\tag{3.24}$$

and for the conformal case,

$$\begin{aligned}\zeta_e^{C'}(0) &= \frac{4}{\beta^2}\zeta_R(3) - \frac{1}{6}\log 2 - \frac{1}{6}\log \beta - 2\zeta'_R(-1) \\ &\quad + \beta^2 \left(\frac{m^2}{96\pi^2} + \frac{7}{5760} + \frac{m^2}{128\pi^2}(\gamma + 2\log 2 - \log \beta) \right) + O(\beta^4).\end{aligned}\tag{3.25}$$

In giving these calculational details we have provided the necessary minimum for anyone who wishes to check our results or who might want to extend them.

4. Results. Thermal interpretation.

We now collect the expressions for the high temperature series of the free energies, $\Phi(\beta)$, of all the fields considered in this paper and these are our main results. They are valid for all m , even and odd.

For the scalar field,

$$\Phi(\beta) \sim -\frac{2}{\beta^3}\zeta_R(3) + \frac{1}{12\beta}\log \beta + \frac{1}{\beta}\zeta'_R(-1) - \frac{1}{2}\left(\frac{m^2}{96\pi^2} + \frac{7}{5760}\right)\beta + O(\beta^3).\tag{4.1}$$

For the conformal (in three dimensions) scalar field,

$$\begin{aligned}\Phi(\beta) \sim & -\frac{2}{\beta^3}\zeta_R(3) + \frac{1}{12\beta}\log\beta + \frac{1}{\beta}\zeta'_R(-1) \\ & - \frac{1}{2}\left(\frac{m^2}{96\pi^2} + \frac{7}{5760} + \frac{m^2}{128\pi^2}(\gamma + 2\log 2 - \log\beta)\right)\beta + O(\beta^3).\end{aligned}\tag{4.2}$$

For the periodic spinor field,

$$\begin{aligned}\Phi(\beta) \sim & \frac{4}{\beta^3}\zeta_R(3) + \frac{1}{3\beta}\log\beta + \frac{4}{\beta}\zeta'_R(-1) + \left(\frac{m^2}{96\pi^2} - \frac{1}{720}\right)\beta \\ & + \frac{m^2\beta}{128\pi^2}(\gamma - \log\beta) + O(\beta^2).\end{aligned}\tag{4.3}$$

For *twisted* (*i.e.* antiperiodic) spinors

$$\Phi(\beta) \sim -\frac{3\zeta_R(3)}{\beta^3} + \frac{1}{3\beta}\log 2 - \frac{\beta}{240} - \frac{m^2\beta}{128\pi^2}\log 2 + O(\beta^2),\tag{4.4}$$

which, as anticipated, has a legitimate thermodynamic sign. Note the absence of $\log\beta/\beta$ and $\zeta'_R(-1)/\beta$ terms. The former means that there is no term proportional to T in the high- T expansion of the internal energy, which is typical of standard spinor thermodynamics in two dimensions.

The unidentified case is obtained by setting m equal to unity.

5. Discussion of results

Our first comment is to remark on the sign of the leading term in the periodic spinor expression, (4.3). This differs from that in [1] which was the result of a basic error. The twisted case, (4.4), exhibits a ‘thermal sign’.

The scalar result, (4.1), also differs from that in [1]; again a consequence of sign errors in that article.

For comparison, the free energy resulting from the small l_3 expansion of the action of the AdS-Taub Bolt instanton, rendered finite by subtraction of the AdS-Taub Nut value, is given by [9],

$$\Phi \sim -\alpha\left(\frac{1}{\beta^3} - \frac{9}{8\pi^2\beta} + \frac{27(m+2)^2}{1024\pi^4}\beta\right)\tag{5.1}$$

where α is a constant.

The precise connection between (5.1) and the quantities evaluated in this paper is problematic. In particular, while the leading term is comparable, there are no transcendental quantities and the dependence on the modding integer, m , is different. Nor does there seem to be any hope of patching up these differences by taking combinations of fields. Of course expression (5.1) is subject to the uncertainties of the regularising subtraction.

6. Conclusion

For conformal scalars and spinors, the awkward form of the eigenvalues seems to necessitate an expansion. This is of no real concern if all we want is the asymptotic behaviour in the extreme oblate case but does prevent a computation of the effective action for *any* value of the squashing parameter, which is something one ought to be able to find. In the scalar field case, for a particular choice of propagation operator, it is possible to calculate the effective action for the complete range of l_3 .

In the course of the calculation several summations arise that have been encountered in other connections. Some extensions of these earlier results are necessary.

The details and manipulations of the spinor calculation may have application in string theory where the calculation of the spectral asymmetry invariant, $\eta(0)$, for twisted spinors on a space whose boundary is a squashed lens space appears [6]. In this case one would also like to evaluate the determinants for the more general periodicity condition (2.5), (2.6).

The entire calculation could be repeated for the arbitrary odd-dimensional sphere by making use of the results of Bär, [7,8], and their extension. In particular it would be relatively straightforward to evaluate the poles of the ζ -function and hence the heat-kernel expansion coefficients in terms of generalised Bernoulli functions. The corresponding questions for the spectral asymmetry function, $\eta(s)$, can also be answered; cf [2] for the three-dimensional, unidentified case.

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